Second Order Calculations of the $O(N)$ σ-Model Laplacian

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Abstract

For slowly varying fields on the scale of the lightest mass the logarithm of the vacuum functional of a massive quantum field theory can be expanded in terms of local functionals satisfying a form of the Schrödinger equation, the principal ingredient of which is a regulated functional Laplacian. We extend a previous work to construct the next to leading order terms of the Laplacian for the Schrödinger equation that acts on such local functionals. Like the leading order the next order is completely determined by imposing rotational invariance in the internal space together with closure of the Poincaré algebra.

1 The $O(N)$ σ-model Laplacian

We are going to study the $O(N)$ σ-model as a test model to construct the Schrödinger representation for non-linear systems. This model has a very interesting structure when it is quantised. Because of its non-linearity it gives characteristics similar to the ones we have in Yang-Mills or the Einstein cases (see [1]).

The $O(N)$ σ-model is defined as the infinite dimensional theory of a particle on an N dimensional sphere parametrised by the function $z^\mu(\sigma, \tau)$ as the variable τ varies. We ask the (σ, τ) plane to be a Minkowski space. The action of the theory has to be a scalar with respect to the Lorentz transformations and the reparametrisations on the sphere. So we can choose it to be

$$S = \frac{1}{2\alpha} \int d\sigma d\tau \, g_{\mu\nu}(\dot{z}^\mu \dot{z}^\nu - \acute{z}^\mu \acute{z}^\nu) \tag{1}$$

where $'$ and $\cdot$ denote differentiations with respect σ and τ respectively, and α is a coupling constant.

We can quantise this model using the Schrödinger Representation (see [3]-[10]) by taking the field $z(\sigma, \tau)$ to be diagonalised at $\tau = 0$ satisfying the relation

$$z^\mu(\sigma, 0)\Psi[z] = z^\mu(\sigma)\Psi[z] \tag{2}$$

for Ψ the Schrödinger wave functional and its conjugate momentum $\pi(\sigma, \tau)$ to be at $\tau = 0$

$$\pi^\nu(\sigma, 0)\Psi[z] = i\alpha \mathbf{D}_\nu(\sigma)\Psi[z] \tag{3}$$

so that the equal time commutation relation

$$[z^\mu(\sigma, 0), \pi_\nu(\sigma', 0)] = i\alpha \delta^\mu_\nu \delta(\sigma, \sigma') \tag{4}$$

is satisfied. In (3) the differential operator is defined with respect to a covariant differentiation whose meaning and structure will be given later on. Though, as z^μ is a 'scalar', $\mathbf{D}_\nu(\sigma)$ takes the usual functional derivative form $\delta/\delta z^\nu(\sigma)$. From (1) we can read the Hamiltonian

$$H = \frac{\alpha}{2}\Delta + \frac{1}{2\alpha} \int d\sigma \, g_{\mu\nu} \acute{z}^\mu \acute{z}^\nu \tag{5}$$

The Laplacian given in (5) as $\Delta = \int d\sigma \, g^{\mu_1\mu_2} \, \mathbf{D}_{\mu_1}(\sigma)\mathbf{D}_{\mu_2}(\sigma) = \int d\sigma_1 d\sigma_2 \, \mathbf{g}^{\mu_1\mu_2}(\sigma_1, \sigma_2)$ $\mathbf{D}_{\mu_1}(\sigma_1) \, \mathbf{D}_{\mu_2}(\sigma_2)$ for $\mathbf{g}^{\mu_1\mu_2}(\sigma_1, \sigma_2) = g^{\mu_1\mu_2}(z(\sigma_1))\delta(\sigma_1 - \sigma_2)$, is not well defined because the two functional derivatives act at the same point σ. Also the determinant of the infinite dimensional metric, $\mathbf{g}$, is ill-defined as the integral on its diagonal $\sigma_1 = \sigma_2$ gives infinity. We can get around this problem by defining the Laplacian to have the regulated expression

$$\Delta_s = \int d\sigma_1 d\sigma_2 \, \mathbf{G}_s^{\mu_1\mu_2}(\sigma_1, \sigma_2) \, \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2). \tag{6}$$

The Kernel $\mathbf{G}$, which takes the place of the infinite dimensional metric $\mathbf{g}$, can be determined by a number of physical requirements. It has been shown in [2] that this is possible at the leading order, when the Laplacian acts on local functionals. We will see here that this is the case also for the next order. Following similar steps with [2] we require that $\mathbf{G}$ is a regularisation of the inverse metric, so we will assume that it depends

on a cut-off parameter, s, with the dimensions of squared length, and takes the form $\mathbf{G}_s^{\mu_1\mu_2}(\sigma_1,\sigma_2) = \mathcal{G}_s(\sigma_1 - \sigma_2)\ K^{\mu_1\mu_2}(\sigma_1,\sigma_2;s)$, where $\mathcal{G}_s(\sigma_1 - \sigma_2) \to \delta(\sigma_1 - \sigma_2)$ as $s \to 0$. K is expandable as a power series in positive integer powers of s, so that it has a finite limit as s goes to zero. Thus $K = \sum_{n=0}^{\infty} K_n s^n$ and $K_0^{\mu_1\mu_2}(\sigma,\sigma) = g^{\mu_1\mu_2}(z(\sigma))$ so that $\lim_{s\to 0}\mathbf{G}^{\mu_1\mu_2}(\sigma_1,\sigma_2) = g^{\mu_1\mu_2}(\sigma_1)\delta(\sigma_1 - \sigma_2)$. To preserve the invariance of the theory under internal rotational symmetry the kernel, $\mathbf{G}$, must be a second rank tensor under the restricted class of co-ordinate transformations of rigid rotations. We can define in addition to $\mathbf{g}^{\mu_1\mu_2}(\sigma_1,\sigma_2)$ its inverse as $\mathbf{g}_{\mu_1\mu_2}(\sigma_1,\sigma_2) = g_{\mu_1\mu_2}(z(\sigma_1))\delta(\sigma_1 - \sigma_2)$ so their contraction gives

$$\int d\sigma \mathbf{g}_{\mu_1\mu}(\sigma_1,\sigma)\mathbf{g}^{\mu\mu_2}(\sigma,\sigma_2) = \delta_{\mu_1}^{\mu_2}\delta(\sigma_1 - \sigma_2) \equiv \mathbf{I}_{\mu_1}^{\mu_2}(\sigma_1,\sigma_2) \tag{7}$$

where $\mathbf{I}_{\mu_1}^{\mu_2}(\sigma_1,\sigma_2)$ is the infinite dimensional Kronecker delta and it is equal to the functional derivative of $z^{\mu_2}(\sigma_2)$ with respect to $z^{\mu_1}(\sigma_1)$, so (4) can be rewritten more compactly as $[z^\mu(\sigma),\pi_\nu(\sigma')] = i\alpha\mathbf{I}_\nu^\mu(\sigma,\sigma')$.

The momentum operator we used in the commutation relation (2) has to be covariant. This property can be carried out to its functional differential operator representation. Given the infinite dimensional metric we can follow the usual construction of the Levi-Civita connection, $\mathbf{D}$, which will transform covariantly under general co-ordinate transformations and therefore under our restricted transformations. Thus if it acts on a scalar will reduce to the usual functional derivative. For the spacial case of an infinite dimensional ultra-local vector $\mathbf{V}^{\mu_1}(\sigma_1)$ we get

$$\mathbf{D}_{\mu_2}(\sigma_2)\mathbf{V}^{\mu_1}(\sigma_1) = \frac{\delta\mathbf{V}^{\mu_1}(\sigma_1)}{\delta z^{\mu_2}(\sigma_2)} + \int d\sigma_3\,\mathbf{\Gamma}_{\mu_2\mu_3}^{\mu_1}(\sigma_1,\sigma_2,\sigma_3)\mathbf{V}^{\mu_3}(\sigma_3) = (D_{\mu_2}V^{\mu_1})|_{z(\sigma_1)}\,\delta(\sigma_1 - \sigma_2)$$
$$\tag{8}$$

where the infinite dimensional Christoffel symbol is related to that on S^N by $\mathbf{\Gamma}_{\mu_2\mu_3}^{\mu_1}(\sigma_1,\sigma_2,\sigma_3) = \delta(\sigma_1 - \sigma_2)\,\delta(\sigma_2 - \sigma_3)\,\Gamma_{\mu_2\mu_3}^{\mu_1}(z(\sigma_1))$ and D is the covariant derivative on S^N. In addition, we can define a finite dimensional intrinsic derivative $\mathcal{D} = \partial/\partial\sigma + z'^\mu D_\mu$, which is a total differential of the σ variable with invariant transformation properties.

Since we work in a Hamiltonian formalism, Poincaré invariance is not manifest and must be imposed by demanding that the generators of these transformations satisfy the Poincaré algebra. Ignoring regularisation the Poincaré generators are the Hamiltonian, given in (5) which generates time τ translations, the momentum $P = \int d\sigma z'^\mu\mathbf{D}_\lambda(\sigma)$ which generates space σ translations and the Lorentz generator $L = -\alpha M + \alpha^{-1}N$, where

$$M = \frac{1}{2}\int d\sigma\,\sigma g^{\mu_1\mu_2}\,\mathbf{D}_{\mu_1}(\sigma)\mathbf{D}_{\mu_2}(\sigma), \quad N = \frac{1}{2}\int d\sigma\,\sigma g_{\mu_1\mu_2}z'^{\mu_1}z'^{\mu_2} \tag{9}$$

which generates Lorentz transformations in the (σ,τ) space. Formally, operators L, H and P satisfy the Poincaré algebra $[P,H] = 0$, $[L,P] = H$, $[L,H] = P$. We require that this algebra holds for the regularised operators. The momentum operator does not need to be regulated. We regulate the Laplacian as in (6) to yield a cut-off Hamiltonian H_s. As seen in [2] the limit $s \to 0$ of the action of H_s on Ψ exists and represents the application of the Hamiltonian on Ψ. Similarly the cut-off dependent Lorentz operator, L_s, should have a finite limit when applied to the physical states. The commutator $[L,P] = H$ implies that L should be regulated with the same kernel as H, so we replace in (9) the operator

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M by

$$\int d\sigma_1 d\sigma_2 \frac{\sigma_1 + \sigma_2}{2} \mathbf{G}_s^{\mu_1\mu_2}(\sigma_1, \sigma_2) \mathbf{D}_{\mu_1}(\sigma_1)\mathbf{D}_{\mu_2}(\sigma_2) \equiv M_s \tag{10}$$

The regularised versions of the Poincaré algebra impose conditions on the Kernel when acting on local functionals named generally F. For example the regularised version of $([L, H] - P)F = 0$ is

$$\frac{1}{4}\left[-\alpha M_s + \alpha^{-1}N, -\alpha\Delta_s + \alpha^{-1}V\right]F = PF \tag{11}$$

We demand that this equation holds order by order in $1/s$ up to order zero. These are the terms that, in the absence of a regulator, involve two functional derivatives at the same point on a single local functional. The terms with positive powers of s will disappear at the limit $s \to 0$. They are equivalent to the $O(s^n)$ terms with $n > 0$ in the small-s expansion (see [11]), which is treated with the re-summation procedure in order to extract the desired zeroth order term. Thus, by requiring $[M_s, \Delta_s]F = 0$, as a restriction for the Kernel, and by ignoring the positive powers of s in the intermediate steps of the calculation we obtain a better approximation for the first term of the small-s expansion. Also we demand $M_s V = 0$ and $\Delta_s N = 0$.

We are interested in constructing the Schrödinger equation for slowly varying fields. This allows us to expand the vacuum-functional in terms of local functionals. For the $O(N)$ σ-model they are integrals of functions of σ, $z(\sigma)$ and a finite number of its derivatives at the point σ. In order to construct the Kernel $\mathbf{G}$, we will consider the conditions that arise from applying the regularised form of the Poincaré algebra to such test functionals. It will be convenient to order them according to the powers of $\mathcal{D}$. If we consider the result of the action of the two functional derivatives from Δ (or M) on a local test functional as a differential operator of σ acting on a delta function, then the order of the operator, that is the highest number of covariant σ derivatives acting on the delta function, depends on the highest number of differentiations on the z's used to construct the local functional. This operator acting on one of the σ arguments of the Kernel, via integration by parts and setting its two arguments equal to each other, with the application of the delta function, will demand the use of more terms of the Kernel expansion with respect to s, depending on the order of the operator and in conclusion on the order of the test functional.

In [2] the least order local functionals were used, which have the form $F_n \equiv \int d\sigma$ $f(z(\sigma), \sigma)_{\mu_1...\mu_n} z'^{\mu_1}...z'^{\mu_n}$ where f is ultra-local. Having in mind the transformation properties (rotation invariance) of $\mathbf{G}$ and its dimension (inverse length) we can set $\mathbf{G}^{\mu\nu}(\sigma, \sigma) = \frac{1}{\sqrt{s}}b_0^0 g^{\mu\nu}$ so that

$$\left((\mathcal{D}|_\sigma + \mathcal{D}|_{\sigma'})\mathbf{G}^{\mu\nu}(\sigma, \sigma')\right)_{\sigma=\sigma'} = \mathcal{D}\left(\frac{b_0^0}{\sqrt{s}}g^{\mu\nu}\right) = 0 \tag{12}$$

up to zeroth order in s and

$$\left(\mathcal{D}|_\sigma \mathcal{D}|_{\sigma'} \mathbf{G}^{\mu\nu}(\sigma, \sigma')\right)_{\sigma=\sigma'} = -\frac{1}{\sqrt{s^3}}(b_0^1 g^{\mu\nu} + s b_1^1 g_{\lambda\rho} z'^\lambda z'^\rho g^{\mu\nu} + s b_2^1 z'^\mu z'^\nu) \tag{13}$$

where b^0_1, b^1_0, b^1_1, b^1_2, ... are dimensionless constants. b^0_0 and b^1_0, given by $b^0_0 = \sqrt{s}\mathcal{G}_s(0)$ and $b^1_0 = \sqrt{s}^3\mathcal{G}''_s(0)$, are determined by our choice of regularisation of the delta-function $\mathcal{G}_s$. As it was shown in [2], these coefficients can be determined by demanding the closure of the Poincaré algebra. This is achieved by taking $b^1_1 = -b^1_2 = -b^0_0/a^2$, with which we can built $\mathbf{G}$ up to first order.

In the same way we assume that for the next order the constraint equations resulting from the closure of the Poincaré algebra, will be independent of the general form of the test functional, as soon as the differential operator is of order four. We can apply the Laplacian on $\int d\sigma\, f(z(\sigma),\sigma)_{\mu\nu}\mathcal{D}z'^{\mu}\mathcal{D}z'^{\nu} \equiv F$, where f is ultra-local. Then, the two functional derivatives in the Laplacian will generate a fourth order differential operator acting on $\delta(\sigma_1 - \sigma_2)$. The consequence of this is that $\Delta_s F$ will depend on the fourth derivative of the Kernel evaluated at co-incident points. Demanding the closure of the Poincaré algebra acting on F will constrain this quantity. F is the lowest order functional of the general form $\int d\sigma f_{\rho_1...\rho_n}\mathcal{D}z'^{\rho_1}...\mathcal{D}z'^{\rho_n}$ that gives a constrain to this order. To simplify the calculations we notice that the part of the result of the action of the Lorentz operator on a general functional, which will contribute to the commutation relations of the Poincaré algebra, is the non-linear one in σ.

We treat F as a scalar so that

$$\mathbf{D}_\mu(\sigma)F = \frac{\delta F}{\delta z^\mu(\sigma)} = D_\mu f_{\rho_1\rho_2}\mathcal{D}z'^{\rho_1}\mathcal{D}z'^{\rho_2} + 2\mathcal{D}^2(f_{\mu\rho_2}\mathcal{D}z'^{\rho_2}) - 2f_{\rho_1\rho_2}z'^{\kappa}z'^{\lambda}R_{\mu\kappa\lambda}{}^{\rho_1}\mathcal{D}z'^{\rho_2} \quad (14)$$

which is an infinite component co-vector. Using the commutators of $\mathbf{D}$ and $\mathcal{D}$, given in [2], we can show that

$$
\begin{aligned}
M_s F &= \int d\sigma\sigma \mathbf{G}^{\mu\nu}(\sigma,\sigma)\Big((D_\mu D_\nu f_{\rho_1\rho_2} - 2R_{\mu\rho_1\nu}{}^{\lambda}f_{\lambda\rho_2})\mathcal{D}z'^{\rho_1}\mathcal{D}z'^{\rho_2} + \\
&\quad 2f_{\rho_1\rho_2}R_{\nu\kappa_1\kappa_2}{}^{\rho_1}R_{\mu\lambda_1\lambda_2}{}^{\rho_2}z'^{\kappa_1}z'^{\kappa_2}z'^{\lambda_1}z'^{\lambda_2} - 4D_\mu f_{\rho_1\rho_2}R_{\nu\kappa\lambda}{}^{\rho_1}z'^{\kappa}z'^{\lambda}\mathcal{D}z'^{\rho_2}\Big) + \\
&\quad 4\int d\sigma\,\sigma\left(\mathcal{D}^2_\sigma\mathbf{G}^{\mu\nu}(\sigma,\sigma')\right)_{\sigma=\sigma'}(D_\nu f_{\mu\rho_2}\mathcal{D}z'^{\rho_2} - f_{\rho_1\mu}R_{\nu\kappa\lambda}{}^{\rho_1}z'^{\kappa}z'^{\lambda}) + \\
&\quad 2\int d\sigma\sigma\left(\mathcal{D}^4_\sigma\mathbf{G}^{\mu\nu}(\sigma,\sigma')\right)_{\sigma=\sigma'}f_{\mu\nu} + 4\int d\sigma\,\mathbf{G}^{\mu\nu}(\sigma,\sigma)f_{\rho_1\rho_2}R^{\rho_1}{}_{\nu\kappa\mu}z'^{\kappa}\mathcal{D}z'^{\rho_2} \quad (15)
\end{aligned}
$$

Using relations (12), (13) and the additional

$$
\begin{aligned}
\left(\mathcal{D}^4_\sigma\mathbf{G}^{\mu\nu}(\sigma,\sigma')\right)_{\sigma=\sigma'} &= \frac{1}{\sqrt{s}^5}\Big(b^2_0\,g^{\mu\nu} + sb^2_1\,g_{\lambda\rho}z'^{\lambda}z'^{\rho}\,g^{\mu\nu} + sb^2_2\,z'^{\mu}z'^{\nu} \\
&\quad + s^2b^2_3\,(g_{\lambda\rho}z'^{\lambda}z'^{\rho})^2\,g^{\mu\nu} + s^2b^2_4\,g_{\lambda\rho}z'^{\lambda}z'^{\rho}\,z'^{\mu}z'^{\nu} + s^2b^2_5\,\mathcal{D}z'^{\mu}\mathcal{D}z'^{\nu} + s^2b^2_6\,g_{\lambda\rho}\mathcal{D}z'^{\lambda}\mathcal{D}z'^{\rho}\,g^{\mu\nu} \\
&\quad + s^2b^2_7\,z'^{(\mu}\mathcal{D}^2z'^{\nu)} + s^2b^2_8\,g_{\lambda\rho}z'^{(\lambda}\mathcal{D}^2z'^{\rho)}g^{\mu\nu}\Big), \quad (16)
\end{aligned}
$$

where $b^2_0 = \sqrt{s}^5\mathcal{G}''''_s(0)$, we can write $M_s F$ in terms of the constants b^i_j. Because of the property

$$\left(\mathcal{D}^k|_\sigma\mathbf{G}^{\mu\nu}(\sigma,\sigma')\right)_{\sigma=\sigma'} = -\left(\mathcal{D}^k|_{\sigma'}\mathbf{G}^{\mu\nu}(\sigma,\sigma')\right)_{\sigma=\sigma'} + O(\sqrt{s}) \quad (17)$$

for k an odd integer, and the symmetry of Δ_s and M_s in σ, σ' there will not be any odd number of intrinsic derivatives acting on $\mathbf{G}$ in our final expressions so we will not need their expansion in terms of b's.

After substituting into (15) we have

$$M_s \int d\sigma f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu = \int d\sigma\, \sigma \Big\{ \frac{2}{\sqrt{s}^5} b_0^2 f_\nu^\nu +$$

$$\frac{1}{\sqrt{s}^3} \big(4b_0^1 D^\nu f_{\nu\mu} \mathcal{D}z'^\mu - 4b_0^1 f_{\mu\nu} R^\mu{}_{\kappa\lambda}{}^\nu z'^\kappa z'^\lambda + 2b_1^2 f_\nu^\nu g_{\kappa\lambda} z'^\kappa z'^\lambda + 2b_2^2 f_{\mu\nu} z'^\mu z'^\nu \big) +$$

$$\frac{1}{\sqrt{s}} \big((J_2 f)_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu + (J_4 f)_{\mu\nu\kappa\lambda} z'^\mu z'^\nu z'^\kappa z'^\lambda + (J_3 f)_{\mu\nu} z'^{(\mu} \mathcal{D}^2 z'^{\nu)} \big) \Big\} +$$

$$\int d\sigma\, \frac{4}{\sqrt{s}} b_0^0 \frac{1-N}{a^2} f_{\mu\nu} z'^\mu \mathcal{D}z'^\nu \tag{18}$$

where

$$(J_2 f)_{\mu\nu} = b_0^0 \tilde{\Delta} f_{\mu\nu} - 2b_0^0 R^\rho_{(\mu} f_{\nu)\rho} + 2b_5^2 f_{\mu\nu} + 2b_6^2 f_\kappa^\kappa g_{\mu\nu} \tag{19}$$

$$(J_3 f)_{\mu\nu} = 2b_7^2 f_{\mu\nu} + 2b_8^2 f_\kappa^\kappa g_{\mu\nu} \tag{20}$$

$$(J_4 f)_{\mu\nu\kappa\lambda} = 2(b_3^2 - \frac{b_0^0}{a^4}) f_\rho^\rho g_{(\mu\nu} g_{\kappa\lambda)} + 2(b_4^2 + \frac{b_0^0}{a^4}) f_{(\mu\nu} g_{\kappa\lambda)} \tag{21}$$

We can easily read of the action of Δ_s from above (term linear in σ). In order to compute the commutator of M_s and Δ_s on $\int f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu$, we need the following relations:

$$\Delta_s \int f_{\kappa\lambda} z'^\kappa \mathcal{D}z'^\lambda = \int d\sigma \Big(\frac{1}{\sqrt{s}^3} 2b_0^1 D^\lambda f_{\kappa\lambda} z'^\kappa +$$

$$\frac{1}{\sqrt{s}} b_0^0 (\tilde{\Delta} f_{\kappa\lambda} + 2\frac{1-N}{a^2} f_{\kappa\lambda}) z'^\kappa \mathcal{D}z'^\lambda \Big) \tag{22}$$

$$M_s \int f_{\kappa\lambda} z'^{(\kappa} \mathcal{D}^2 z'^{\lambda)} = \ldots + \int d\sigma \Big\{ \frac{3}{\sqrt{s}^3} b_0^1 D^\lambda f_{\kappa\lambda} z'^\kappa +$$

$$\frac{b_0^0}{\sqrt{s}} \big(D^\kappa f_{\kappa\lambda} \mathcal{D}^2 z'^\lambda - R^\nu{}_{\gamma\rho}{}^\lambda D_\nu f_{\kappa\lambda} z'^\kappa z'^\rho z'^\gamma +$$

$$\frac{3}{a^2} f_\kappa^\kappa g_{\gamma\rho} z'^\gamma \mathcal{D}z'^\rho - \frac{3N}{a^2} f_{\kappa\lambda} z'^\kappa \mathcal{D}z'^\lambda \big) \Big\} \tag{23}$$

$$M_s \int f_\kappa \mathcal{D}z'^\kappa = \ldots - \int d\sigma \frac{2}{\sqrt{s}} b_0^0 R^\kappa_\lambda f_\kappa z'^\lambda \tag{24}$$

$$M_s \int f_{\kappa\lambda} z'^\kappa z'^\lambda = \ldots + \int d\sigma \frac{2}{\sqrt{s}} b_0^0 D^\kappa f_{\kappa\lambda} z'^\lambda \tag{25}$$

where ... represents the linear in σ part of the action of M_s on the specific functional. Using relations (22)-(25) we derive the action of the operators $M_s \Delta_s$ and $\Delta_s M_s$ on $\int f_{\mu\nu} \mathcal{D}z'^\mu \mathcal{D}z'^\nu$

to be

$$M_s \Delta_s \int f_{\mu\nu} \mathcal{D}z'^{\mu} \mathcal{D}z'^{\nu} =$$

$$... + \int d\sigma \Big\{ \frac{1}{s} \Big[4b_0^0 \frac{1-N}{a^2} \Big(b_0^0 \tilde{\Delta} f_{\mu\nu} - 2b_0^0 R_{\mu}^{\rho} f_{\nu\rho} + 2b_5^2 f_{\mu\nu} + 2b_6^2 f_{\rho}^{\rho} g_{\mu\nu} \Big) z'^{\mu} \mathcal{D}z'^{\nu} +$$

$$4b_0^0 D^{\mu} \Big(2(b_3^2 - \frac{b_0^0}{a^4}) f_{\rho}^{\rho} g_{(\mu\nu} g_{\kappa\lambda)} + 2(b_4^2 + \frac{b_0^0}{a^4}) f_{(\mu\nu} g_{\kappa\lambda)} \Big) z'^{\nu} z'^{\kappa} z'^{\lambda}$$

$$b_0^0 D^{\nu} (J_3 f)_{\nu\lambda} \mathcal{D}^2 z'^{\lambda} - b_0^0 R^{\nu}_{\ \gamma\rho}{}^{\lambda} D_{\nu} (J_3 f)_{\kappa\lambda} z'^{\kappa} z'^{\rho} z'^{\gamma} +$$

$$\frac{3b_0^0}{a^2} (J_3 f)^{\nu}_{\nu} g_{\gamma\rho} z'^{\gamma} \mathcal{D}z'^{\rho} - \frac{3Nb_0^0}{a^2} (J_3 f)_{\gamma\rho} z'^{\gamma} \mathcal{D}z'^{\rho} \Big]$$

$$\frac{1}{s^2} \Big(4b_0^1 b_0^0 2 \frac{1-N}{a^2} D^{\nu} f_{\nu\rho} + 4b_0^0 b_1^2 D_{\rho} f_{\kappa}^{\kappa} + 4b_0^0 b_2^2 D^{\mu} f_{\mu\rho} +$$

$$6b_0^1 b_7^2 D^{\nu} f_{\nu\rho} + 6b_0^1 b_8^2 D_{\rho} f_{\kappa}^{\kappa} + 4 \frac{b_0^1 b_0^0}{a^2} (2 D_{\rho} f_{\kappa}^{\kappa} - 2 D^{\nu} f_{\nu\rho}) \Big) z'^{\rho} \Big\} \tag{26}$$

and

$$\Delta_s M_s \int f_{\mu\nu} \mathcal{D}z'^{\mu} \mathcal{D}z'^{\nu} = ... + \int d\sigma \Big\{ \frac{4}{s} b_0^{0 2} \frac{1-N}{a^2} (\tilde{\Delta} f_{\mu\nu} + 2 \frac{1-N}{a^2} f_{\mu\nu}) z'^{\mu} \mathcal{D}z'^{\nu}$$

$$+ \frac{4}{s^2} b_0^0 \frac{1-N}{a^2} 2 b_0^1 D^{\nu} f_{\mu\nu} z'^{\mu} \Big\} \tag{27}$$

The $O(1/s^3)$ order does not contribute to the commutator. The $O(1/s^2)$ order in (26) and (27) can be factorised with coefficients the various z combinations which result the following relations

$$- 4b_0^1 b_0^0 \frac{1}{a^2} + 2b_0^0 b_2^2 + 3b_0^1 b_7^2 = 0, \quad 4 b_0^1 b_0^0 \frac{1}{a^2} + 2b_0^0 b_1^2 + 3b_0^1 b_8^2 = 0 \tag{28}$$

The $O(1/s)$ order gives $b_3^2 = -b_4^2 = b_0^0/a^4$ and $b_5^2 = b_6^2 = b_7^2 = b_8^2 = 0$, so that (28) becomes $-2b_0^1/a^2 + b_2^2 = 0$ and $2 b_0^1/a^2 + b_1^2 = 0$. From these relations the b's needed up to this order in the Kernel are completely determined. Assuming that the theory is consistent these coefficients are the same for every test functional we use with the same highest number of σ derivatives, acting on z, within F. We can write K as a Taylor expansion of the σ variables with the help of $\mathbf{W}^{\mu_1}_{\mu_2}(\sigma_1, \sigma_2)$, defined by $\mathcal{D}|_{\sigma_1} \mathbf{W}^{\mu_1}_{\mu_2}(\sigma_1, \sigma_2) = 0$, and $\mathbf{W}^{\mu_1}_{\mu_2}(\sigma, \sigma) = \delta^{\mu_1}_{\mu_2}$ (see [2]). Up to order $O((\sigma_1 - \sigma_2)^5)$ we get

$$K^{\mu_1 \mu_2}(\sigma_1, \sigma_2) = \mathbf{W}^{\mu_1}_{\nu}(\sigma_1, \sigma_2) \Big(g^{\nu\mu_2}|_{z(\sigma_2)} - \frac{1}{2}(\sigma_1 - \sigma_2)^2 R^{\nu}_{\ \lambda}{}^{\mu_2}_{\ \rho} z'^{\lambda} z'^{\rho}|_{z(\sigma_2)} +$$

$$\frac{1}{4!(N-1)} (\sigma_1 - \sigma_2)^4 R_{\kappa\sigma} z'^{\kappa} z'^{\sigma} R^{\nu}_{\ \lambda}{}^{\mu_2}_{\ \rho} z'^{\lambda} z'^{\rho}|_{z(\sigma_2)} +$$

$$\frac{b_0^1}{6 b_0^0} \frac{1}{s} (\sigma_1 - \sigma_2)^4 R^{\nu}_{\ \lambda}{}^{\mu_2}_{\ \rho} z'^{\lambda} z'^{\rho}|_{z(\sigma_2)} \Big) \tag{29}$$

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While up to order $O\left((\sigma_1 - \sigma_2)^3\right)$, K is s independent, an s dependent term appears in the next order. In a similar way we can determine more terms of the expansion by asking the closure of the Poincaré algebra when acting on test functionals of higher order.

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